WRITTEN EXAM

ALGEBRAIC STRUCTURES (2020/21)

Date: 14/06/2021

The maximum number of points you can score is 10. You get 1 point for free.

Instructions: You should work by yourself to solve these problems. You can consult the Lecture Notes and the slides of the course. You should not consult other sources. You should give references to each result of the Lecture Notes used in your answers.

Read the questions carefully and take your time to access which part of the theory you need for each exercise. All your answers should be accompanied by a justification.

Good luck!

Exercise 1. [1.5 point] *Which of the following polynomials is irreducible?*

a) $g = 7X^3 + 14X^2 + 21X - 2$ *over* $\mathbb{Q}[X]$.

b) $h = (1+i)X^{25} + 62$ *over* $\mathbb{Q}(i)[X]$.

Don't forget to justify your answer.

Exercise 2. [1.5 point] *Find a prime decomposition of* $\alpha = 4 + 2i$ *in Let* $\mathbb{Z}[i]$. *Justify your answer by proving that the elements provided in your decomposition are indeed prime in* $\mathbb{Z}[i]$.

Exercise 3. [1.5 point] *Let* $p \in \mathbb{Z}$ *be a prime number and* $\mathbb{Z}[p^{-1}] := \{\frac{a}{b} \in \mathbb{Q}; a \in \mathbb{Z}, b = p^n$ *for some* $n \in \mathbb{Z}_{\geq 0}\}$. *Show that a prime* $q \in \mathbb{Z}$ *is a prime element of* $\mathbb{Z}[p^{-1}]$ *if, and only if,* $\gcd(p,q) = 1$. *Don't forget to justify your answer.*

Exercise 4. [1.5 point] *Let* K *be a field. Let* α *be a zero of* $f = X^2 - 12$ *and* β *a zero of* $g = X^2 - 13$. *Determine if* $K(\alpha) \simeq K(\beta)$ *in the following cases:*

a) $K = \mathbb{Q}$.

b) $K = \mathbb{F}_5$.

Don't forget to justify your answer.

Exercise 5. [3 points] *Let* $\varphi : \mathbb{Z}[X]/(X^2+1) \to \mathbb{F}_{25}$ *given by* $\sum a_i X^i \mapsto \sum 2^i \overline{a_i}$ *where* $\overline{a_i}$ *is the residue of* a_i *modulo 5.*

a) [1.5 point] *Show that* φ *is a ring homomorphism and determine its image. Don't forget to justify your answer. When showing that the map is a homomorphism, you should also show that it is well defined.*

b) [1.5 point] *Determine* $\ker(\varphi)$. *Is* $\ker(\varphi)$ *a prime ideal? Is it maximal? Don't forget to justify your answer.*